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An approximation of American option prices in a jump-diffusion model

Sabrina Mulinacci*

Dipartimento di Matematica, Università di Pisa, Via F. Buonarroti 2, 56100 Pisa, Italy

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Abstract

In this paper, an effectively computable approximation of the price of an American option in a jump-diffusion market model will be shown: results of convergence in L^p and a.s. will be proved.

Keywords: American option pricing; Convergence; Jump-diffusion; Snell envelope

0. Introduction

The problem of computation of the price of an American option in the Black–Scholes case has been examined closely: an exact formula is known (see Lamberton and Lapeyre, 1991, p. 80) but it is impossible to compute it exactly and we need numerical methods of approximation. Among the vast literature on this subject we mention Barone–Adesi and Whaley (1987), Bensoussan and Lions (1978), Cox and Rubinstein (1985), and Jaillet et al. (1990). In this paper we consider a market with discontinuous prices as, for example, in Jeanblanc–Picqué and Pontier (1990). It is a complete model for which it is known that the American option price is the Snell envelope of the option. In the same way as in the paper of Mercurio and Runggaldier (1993), who obtained a computable approximation of the value of an European option, in this article a discrete-time approximation of the market is presented: it will be shown that, in this case, an explicit formula for the Snell envelope can be obtained and results of convergence to the original American option price will be introduced. At the end, in the particular case of the American put, a really computable formula will be presented.

* E-mail: mulinacc@dm.unipi.it.

1. The model

Let us consider a market where three assets are traded and let us assume that there is a fixed time horizon $0 < T < \infty$. The first asset is a bond whose price evolves according to the differential equation

$$\begin{aligned} dS_0(t) &= S_0(t)r(t)dt, \quad 0 \leq t \leq T, \\ S_0(0) &= 1. \end{aligned} \quad (1.1)$$

The remaining two assets are “risky” and their prices are modelled by the following linear stochastic equations for $i = 1, 2$:

$$\begin{aligned} dS_i(t) &= S_i(t^-) \{ b_i(t)dt + \sigma_i(t)dW(t) + \phi_i(t)dN(t) \}, \\ S_i(0) &= S_i^0, \quad S_i^0 \text{ deterministic.} \end{aligned} \quad (1.2)$$

More precisely, the process $W = \{W(t), \mathcal{F}_t^W; 0 \leq t \leq T\}$ is a standard Wiener process on the space $(\Omega^W, \mathcal{F}^W, \mathbb{P}^W)$ and the filtration \mathcal{F}^W is the augmentation under \mathbb{P}^W of the filtration generated by W . The process $N = \{N(t), \mathcal{F}_t^N; 0 \leq t \leq T\}$ is a Poisson process on the space $(\Omega^N, \mathcal{F}^N, \mathbb{P}^N)$ with intensity $\lambda(t)$ and the filtration \mathcal{F}^N is the augmentation under \mathbb{P}^N of the filtration generated by N .

Let us consider the space $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega^W \times \Omega^N, \mathcal{F}^W \otimes \mathcal{F}^N, \mathbb{P}^W \times \mathbb{P}^N)$, where W and N are independent.

We assume $\lambda(t)$ to be deterministic and bounded, while the $b_i(t)$, $\sigma_i(t)$, $\phi_i(t)$ and $r(t)$ are supposed deterministic and continuous.

We suppose $1 + \phi_i(t) \geq \delta > 0$, for $t \in [0, T]$ and $i = 1, 2$.

If

$$R(t) = \exp \left\{ - \int_0^t r(s)ds \right\},$$

we have

$$d(R(t)S_i(t)) = R(t)S_i(t^-) \{ (b_i(t) - r(t))dt + \sigma_i(t)dW(t) + \phi_i(t)dN(t) \},$$

where $R(t)S_i(t)$ are the discounted prices of the risky assets. We put $\tilde{S}_i(t) = R(t)S_i(t)$.

As in Mercurio and Runggaldier (1993), suppose that, for $t \in [0, T]$, we have

$$\begin{aligned} |\sigma_1(t)\phi_2(t) - \sigma_2(t)\phi_1(t)| &\geq \delta > 0, \\ \frac{(b_2(t) - r(t))\sigma_1(t) - (b_1(t) - r(t))\sigma_2(t)}{\sigma_2(t)\phi_1(t) - \sigma_1(t)\phi_2(t)} &> 0. \end{aligned}$$

Under these assumptions it can be shown (see Jeanblanc–Picqué and Pontier, 1990) that there exists a unique equivalent martingale measure \mathbb{P}^* whose Radon–Nikodym derivative with respect to \mathbb{P} satisfies

$$dL(t) = L(t^-) \{ -\theta(t)dW(t) + (p(t) - 1)(dN(t) - \lambda(t)dt) \},$$

with

$$\theta(t) = \frac{(b_2(t) - r(t))\phi_1(t) - (b_1(t) - r(t))\phi_2(t)}{\sigma_2(t)\phi_1(t) - \sigma_1(t)\phi_2(t)} > 0, \quad (1.3)$$

$$p(t)\lambda(t) = \frac{(b_2(t) - r(t))\sigma_1(t) - (b_1(t) - r(t))\sigma_2(t)}{\sigma_2(t)\phi_1(t) - \sigma_1(t)\phi_2(t)} > 0. \quad (1.4)$$

Furthermore, $\mathbb{P}^* = \mathbb{P}_W^* \times \mathbb{P}_N^*$, where \mathbb{P}_W^* and \mathbb{P}_N^* have Radon–Nikodym derivatives $L^W(T)$ and $L^N(T)$ with respect to \mathbb{P}^W and \mathbb{P}^N which satisfy

$$\begin{aligned} dL^W(t) &= -L^W(t^-)\theta(t)dW(t), \\ dL^N(t) &= L^N(t^-)(p(t) - 1)(dN(t) - \lambda(t)dt) \end{aligned}$$

and

$$L(t) = L^W(t)L^N(t).$$

The process

$$W^*(t) = W(t) + \int_0^t \theta(s)ds$$

is a $(\mathcal{F}_t, \mathbb{P}^*)$ -standard Wiener process and $N(t)$ is a $(\mathcal{F}_t, \mathbb{P}^*)$ -Poisson process with intensity $p(t)\lambda(t)$.

The prices $\tilde{S}_i(t)$ are \mathbb{P}^* -martingales and satisfy

$$d\tilde{S}_i(t) = \tilde{S}_i(t^-)\{\sigma_i(t)dW^*(t) + \phi_i(t)(dN(t) - p(t)\lambda(t)dt)\} \quad (1.5)$$

and the market is complete.

By (1.5) we have

$$\begin{aligned} \tilde{S}_i(t) &= S_i(0) \exp \left\{ \int_0^t \left(-\phi_i(s)p(s)\lambda(s) - \frac{1}{2}\sigma_i^2(s) \right) ds \right. \\ &\quad \left. + \int_0^t \sigma_i(s) dW^*(s) + \int_0^t f_i(s) dN(s) \right\} \end{aligned} \quad (1.6)$$

with $f_i(s) = \log(1 + \phi_i(s))$.

Due to the boundedness of all coefficients, it can be shown that

$$\sup_{0 \leq t \leq T} |\tilde{S}_i(t)| \in L^p(\mathbb{P}^*) \quad \text{for every } p \in [1, +\infty) \quad (1.7)$$

see for example Lemma III.2.1 of Xue (1992).

We consider an American option $\psi(S_1(t))$ with $\psi: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \cup \{0\}$, uniformly continuous and such that $|\psi(x)| \leq Ax + B$ with positive A and B (in the case of a put $\psi(x) = (K - x)^+$ and in the case of a call $\psi(x) = (x - K)^+$).

Since we are in a complete market, as in the Black–Scholes case, the price of the option at a time t is (see Geske, 1979a, p. 80)

$$U(t) = \text{ess sup}_{\tau \in \mathcal{F}_{t,T}} E^* \left[e^{-\int_t^\tau r(s)ds} \psi(S_1(\tau)) | \mathcal{F}_t \right], \quad (1.8)$$

where $\mathcal{T}_{t,T}$ is the set of stopping times with respect to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ and with values in $[t, T]$.

Remark 1.1 Some authors suggest to work with a market with just one asset and to use $\hat{\mathbb{P}}$, the *minimal equivalent martingale measure* (see Föllmer and Schweizer, 1991, for a definition of $\hat{\mathbb{P}}$), instead of the equivalent martingale probability \mathbb{P}^* . Runggaldier and Schweizer (1993) give a formula for $d\hat{\mathbb{P}}/d\mathbb{P}$ (under appropriate conditions) and show that, under $\hat{\mathbb{P}}$, $\tilde{S}_1(t)$ satisfies

$$d\tilde{S}_1(t) = \tilde{S}_1(t^-) \{ \sigma_1(t) d\hat{W}(t) + \phi_1(t) (dN(t) - \hat{\lambda}(t) dt) \},$$

where $\hat{W}(t)$ is a Wiener process and $N(t)$ is an independent Poisson process with intensity

$$\hat{\lambda}(t) = \lambda(t) \left[1 - \phi_1(t) \frac{b_1(t) + \phi_1(t)\lambda(t)}{\sigma_1^2(t) + \phi_1^2(t)\lambda(t)} \right].$$

If we use $\hat{\mathbb{P}}$ instead of \mathbb{P}^* , we obtain results analogous to the ones obtained in the following sections.

2. Discrete model

In this section we define a discrete process, which, as we will see in Sections 3 and 4, will allow us to obtain approximations of the price $U(t)$ of an American option.

Consider a particular class of models (1.1) and (1.2) with $\sigma_1(t)$ and $\phi_1(t)$ piecewise constant. More precisely, given a positive integer N , we consider an equispaced subdivision of $[0, T]$, $0 = t_0 < t_1 < \dots < t_N = T$, with $t_j - t_{j-1} = \Delta = T/N$, and let

$$\sigma_1(t) = \sigma_0 I_{\{0\}}(t) + \sum_{j=1}^N \sigma_j I_{(t_{j-1}, t_j]}(t),$$

$$\phi_1(t) = \phi_0 I_{\{0\}}(t) + \sum_{j=1}^N \phi_j I_{(t_{j-1}, t_j]}(t)$$

and $f_j = \log(1 + \phi_j)$ for $j = 0, \dots, N$.

In this way we obtain from (1.6) that the discounted price of the first risky asset, evaluated on the points t_j of the partition, is given by the recursive formula

$$\tilde{S}_1(t_j) = \tilde{S}_1(t_{j-1}) \exp \left\{ \int_{t_{j-1}}^{t_j} -\phi_j p(s) \lambda(s) ds - \frac{1}{2} \sigma_j^2 \Delta + \sigma_j \Delta W_j^* + f_j \Delta N_j \right\},$$

where

$$\Delta W_j^* = W^*(t_j) - W^*(t_{j-1}) \sim \mathcal{N}(0, \Delta)$$

and ΔN_j are random independent Poisson variables with parameters

$$\Delta_j = \int_{t_{j-1}}^{t_j} \lambda(s) p(s) ds.$$

Let

$$\alpha_j = \int_{t_{j-1}}^{t_j} -\phi_j p(s) \lambda(s) ds \quad \beta_j = \int_{t_{j-1}}^{t_j} r(s) ds.$$

Since $R(t_{j-1})/R(t_j) = e^{\beta_j}$, we have

$$\tilde{S}_1(t_j) = S_1(t_{j-1}) \exp \{ \alpha_j + \beta_j - \frac{1}{2} \sigma_j^2 \Delta + \sigma_j \Delta W_j^* + f_j \Delta N_j \}. \quad (2.1)$$

We consider the problem of explicit computation of the Snell envelope \tilde{U}_j of the discrete process $R(t_j)\psi(S_1(t_j))$ with $j = 0, \dots, N$, where ψ is the function introduced in Section 1,

$$\tilde{U}_j = \operatorname{ess\,sup}_{\tau^N \in \mathcal{T}_{t_j, T}^N} E^*[R(\tau^N)\psi(S_1(\tau^N)) | \mathcal{F}_j],$$

where $\mathcal{T}_{t_j, T}^N$ is the set of stopping times with respect to the filtration $(\mathcal{F}_j)_{j=0,1,\dots,N}$ with $\mathcal{F}_j = \mathcal{F}_{t_j}$ and with values in $\{t_j, t_{j+1}, \dots, t_N\}$.

We observe that $S_1(t_j)$ with $j = 0, \dots, N$ is a Markov chain with respect to the space $(\Omega, \mathcal{F}, (\mathcal{F}_j)_{j=0,1,\dots,N}, \mathbb{P}^*)$, whose transition probability P_j satisfies, for every function $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$,

$$E^*[f(S_1(t_j)) | \mathcal{F}_{j-1}] = P_j f(S_1(t_{j-1})) \quad \text{for } j = 1, \dots, N. \quad (2.2)$$

From (2.1) and (2.2), since $S_1(t_{j-1})$ is \mathcal{F}_{j-1} -measurable and $B = \exp \{ \alpha_j + \beta_j - \frac{1}{2} \sigma_j^2 \Delta + \sigma_j \Delta W_j^* + f_j \Delta N_j \}$ is independent of \mathcal{F}_{j-1} , we have

$$E^*[f(S_1(t_j)) | \mathcal{F}_{j-1}] = E^*[f(S_1(t_{j-1})B) | \mathcal{F}_{j-1}] = P_j f(S_1(t_{j-1})),$$

with $P_j f(z) = E^*[f(zB)]$.

Let $A = z \exp \{ \alpha_j + \beta_j - \frac{1}{2} \sigma_j^2 \Delta + \sigma_j \Delta W_j^* \}$. Since $\mathbb{P}^* = \mathbb{P}_W^* \times \mathbb{P}_N^*$, using Fubini's theorem, we have

$$\begin{aligned} P_j f(z) &= E^*[f(A \exp \{ f_j \Delta N_j \})] \\ &= E_1^W [E_1^N [f(A \exp \{ f_j \Delta N_j \})]] \\ &= E_1^W \left[\sum_{h \in \mathbb{N}} f(A \exp \{ f_j h \}) e^{-\Lambda_j} \frac{\Lambda_j^h}{h!} \right]. \end{aligned}$$

But $f(A \exp \{ f_j h \}) e^{-\Lambda_j} \Lambda_j^h / h! \geq 0$ for every $h \in \mathbb{N}$, then

$$\begin{aligned} P_j f(z) &= e^{-\Lambda_j} \sum_{h \in \mathbb{N}} \frac{\Lambda_j^h}{h!} E_1^W [f(A \exp \{ f_j h \})] \\ &= e^{-\Lambda_j} \sum_{h \in \mathbb{N}} \frac{\Lambda_j^h}{h!} \int_{-\infty}^{+\infty} f(z \exp \{ \alpha_j + \beta_j - \frac{1}{2} \sigma_j^2 \Delta \\ &\quad + \sigma_j \sqrt{\Delta} g + f_j h \}) \frac{e^{-g^2/2}}{\sqrt{2\pi}} dg. \end{aligned}$$

It is well-known that the Snell envelope of a discrete time process $(Z_n)_{0 \leq n \leq N}$ is of the form

$$U_N = Z_N, \quad U_n = \max \{Z_n, E[U_{n+1} | \mathcal{F}_n]\};$$

then the Snell envelope of $R(t_j)\psi(S_1(t_j))$ is of the form $\tilde{U}_j = u(j, S_1(t_j))$, where u is defined by

$$\begin{aligned} u(N, \cdot) &= R(T)\psi(\cdot), \\ u(j-1, \cdot) &= \max \{R(t_{j-1})\psi(\cdot); P_j u(j, \cdot)\}, \quad j = 1, \dots, N. \end{aligned} \quad (2.3)$$

Thus we obtain the following.

Proposition 2.1. *The Snell envelope of $R(t_j)\psi(S_1(t_j))$ is the process \tilde{U}_j given by*

$$\begin{aligned} \tilde{U}_N &= R(T)\psi(S_1(T)), \\ \tilde{U}_{j-1} &= \max \{R(t_{j-1})\psi(S_1(t_{j-1})); P_j u(j, S_1(t_{j-1}))\}, \quad j = 1, \dots, N, \end{aligned}$$

where

$$\begin{aligned} P_j u(j, z) &= e^{-\Delta_j} \sum_{h \in \mathbb{N}} \frac{\Delta_j^h}{h!} \int_{-\infty}^{+\infty} u(j, z \exp\{\alpha_j + \beta_j - \tfrac{1}{2}\sigma_j^2 \Delta \\ &\quad + \sigma_j \sqrt{\Delta} g + f_j h\}) \frac{e^{-g^2/2}}{\sqrt{2\pi}} dg. \end{aligned} \quad (2.4)$$

Remark 2.1. We will see that, in the case of an American put option, this explicit formula is really computable using numerical methods.

3. L^p approximations of the prices

3.1. Continuous case

We consider a particular class of models (1.1) and (1.2) in which some of the coefficients are piecewise constant and approximate those of the original model. We will see in which way the respective American option prices converge to the original model price.

We assume m to be a positive integer and \mathcal{P}^m to be an equispaced partition of $[0, T]$:

$$\mathcal{P}^m = \{(t_0, t_1, \dots, t_{2^m}): 0 = t_0 < t_1 < \dots < t_{2^m} = T, t_j - t_{j-1} = \Delta = T/2^m\}.$$

Let

$$\sigma^m(t) = \sigma_1(0)I_{\{0\}}(t) + \sum_{j=1}^{2^m} \sigma(t_{j-1})I_{(t_{j-1}, t_j]}(t), \quad (3.1)$$

$$\phi^m(t) = \phi_1(0)I_{\{0\}}(t) + \sum_{j=1}^{2^m} \phi(t_{j-1})I_{(t_{j-1}, t_j]}(t). \quad (3.2)$$

By the continuity of $\sigma_1(t)$ and $\phi_1(t)$, we have

$$\lim_{m \rightarrow +\infty} \sigma^m(t) = \sigma_1(t) \quad \text{and} \quad \lim_{m \rightarrow +\infty} \phi^m(t) = \phi_1(t)$$

uniformly with respect to t .

Given the equivalent martingale measure \mathbb{P}^* , defined in Section 1, let us consider a sequence of risky assets $S^m(t)$, which are approximations of $S_1(t)$, with discounted prices $\tilde{S}^m(t) = R(t)S^m(t)$ which satisfy (1.5) with $i = 1$ and with $\sigma_1(t) = \sigma^m(t)$, $\phi_1(t) = \phi^m(t)$, i.e.

$$\tilde{S}^m(t) = \tilde{S}^m(t^-) \{ \sigma^m(t) dW^*(t) + \phi^m(t) (dN(t) - p(t)\lambda(t)dt) \}. \quad (3.3)$$

It can be shown, in the same way as (1.7), that

$$\sup_{m \in \mathbb{N}} E^* \left[\sup_{0 \leq t \leq T} |\tilde{S}^m(t)|^p \right] < +\infty \quad (3.4)$$

for every $p \in [1, +\infty)$.

If we consider the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the process $S^m(t)$ satisfies

$$dS^m(t) = S^m(t^-) \{ b^m(t)dt + \sigma^m(t)dW(t) + \phi^m(t)dN(t) \}, \quad (3.5)$$

with $b^m(t) = r(t) + \sigma^m(t)\theta(t) - \phi^m(t)p(t)\lambda(t)$, where $\theta(t)$ and $p(t)\lambda(t)$ are the same as in (1.3) and (1.4).

Notice that, if instead of the original triplet $(S_0(t), S_1(t), S_2(t))$, we consider the triplet $(S_0(t), S^m(t), S_2(t))$, the assumptions of Section 1 still hold for sufficiently large m and, proceeding as in Section 1, and referring to the results of Jeanblanc–Picqué and Pontier (1990), we obtain the same equivalent martingale measure \mathbb{P}^* and the completeness of the market. Thus, the price of the American option $\psi(S^m(t))$ is

$$U^m(t) = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} E^* \left[e^{-\int_t^\tau r(s)ds} \psi(S^m(\tau)) \mid \mathcal{F}_t \right], \quad (3.6)$$

where $\mathcal{T}_{t,T}$ is the set of stopping times introduced in Section 1.

We shall prove the following:

Theorem 3.1. *Let $U^m(t)$ be the sequence of prices of the American options $\psi(S^m(t))$ and $U(t)$ be the price of the American option $\psi(S_1(t))$ (see (1.8)) at a time t : for every t , $U^m(t)$ converges in $L^1(\mathbb{P}^*)$ to $U(t)$ (uniformly with respect to t).*

The next result will be useful in the proof of Theorem 3.1.

Proposition 3.2. *The paths of $S^m(t)$ converge to the paths of $S_1(t)$ uniformly in probability.*

This means that $\sup_{0 \leq s \leq T} |S^m(s) - S_1(s)|$ converges to zero in probability \mathbb{P}^* .

Proof of Proposition 3.2. Let $N^*(s)$ be the compensated Poisson process $N^*(s) = N(s) - \lambda(s)p(s)$.

If

$$A(s) = r(s) - \phi_1(s)p(s)\lambda(s) - \frac{1}{2}\sigma_1^2(s) + f_1(s)p(s)\lambda(s),$$

$$A^m(s) = r(s) - \phi^m(s)p(s)\lambda(s) - \frac{1}{2}(\sigma^m(s))^2 + f^m(s)p(s)\lambda(s),$$

$$B(t) = \int_0^t A(s)ds + \int_0^t \sigma_1(s)dW^*(s) + \int_0^t f_1(s)dN^*(s)$$

and

$$B^m(t) = \int_0^t A^m(s)ds + \int_0^t \sigma^m(s)dW^*(s) + \int_0^t f^m(s)dN^*(s),$$

then

$$S_1(t) = S_1(0)e^{B(t)}$$

and

$$S^m(t) = S_1(0)e^{B^m(t)}.$$

It can easily be deduced that

$$\begin{aligned} & \sup_{0 \leq t \leq T} |S^m(t) - S_1(t)| \\ & \leq S_1(0) \sup_{0 \leq t \leq T} [\max\{e^{B(t)}; e^{B^m(t)}\}] \sup_{0 \leq t \leq T} |B^m(t) - B(t)| \end{aligned}$$

and using Hölder's inequality, (1.7) and (3.4) we obtain

$$E^* \left[\sup_{0 \leq t \leq T} |S^m(t) - S_1(t)| \right] \leq c E^* \left[\sup_{0 \leq t \leq T} |B^m(t) - B(t)|^2 \right]^{1/2},$$

where c is a constant.

But

$$\begin{aligned} & E^* \left[\sup_{0 \leq t \leq T} |B^m(t) - B(t)|^2 \right] \\ & \leq 3 \left(\int_0^T |A(s) - A^m(s)| ds \right)^2 + 12 \int_0^T |\sigma_1(s) - \sigma^m(s)|^2 ds \\ & \quad + 12 \int_0^T |f_1(s) - f^m(s)|^2 p(s)\lambda(s) ds, \end{aligned}$$

thus we get

$$\begin{aligned}
 & E^* \left[\sup_{0 \leq t \leq T} |S^m(t) - S_1(t)| \right] \\
 & \leq c \left\{ 3 \left(\int_0^T |A(s) - A^m(s)| ds \right)^2 + 12 \int_0^T |\sigma_1(s) - \sigma^m(s)|^2 ds \right. \\
 & \quad \left. + 12 \int_0^T |f_1(s) - f^m(s)|^2 p(s) \lambda(s) ds \right\}^{1/2}, \tag{3.7}
 \end{aligned}$$

Since $A^m(s)$, $\sigma^m(s)$ and $f^m(s)$ converge uniformly respectively to $A(s)$, $\sigma_1(s)$ and $f_1(s)$, we conclude the proof. \square

We are now able to prove Theorem 3.1.

Proof of Theorem 3.1. For every $\tau \in \mathcal{T}_{t,T}$, we have a.s.

$$\begin{aligned}
 & E^* \left[e^{-\int_t^\tau r(s) ds} \psi(S^m(\tau)) | \mathcal{F}_t \right] \\
 & \leq E^* \left[e^{-\int_t^\tau r(s) ds} \psi(S_1(\tau)) | \mathcal{F}_t \right] \\
 & \quad + E^* \left[e^{-\int_t^\tau r(s) ds} |\psi(S^m(\tau)) - \psi(S_1(\tau))| | \mathcal{F}_t \right] \\
 & \leq \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} E^* \left[e^{-\int_t^\tau r(s) ds} \psi(S_1(\tau)) | \mathcal{F}_t \right] \\
 & \quad + \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} E^* \left[e^{-\int_t^\tau r(s) ds} |\psi(S^m(\tau)) - \psi(S_1(\tau))| | \mathcal{F}_t \right],
 \end{aligned}$$

from which we get

$$U^m(t) \leq U(t) + \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} E^* \left[e^{-\int_t^\tau r(s) ds} |\psi(S^m(\tau)) - \psi(S_1(\tau))| | \mathcal{F}_t \right].$$

In the same way we have

$$U(t) \leq U^m(t) + \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} E^* \left[e^{-\int_t^\tau r(s) ds} |\psi(S^m(\tau)) - \psi(S_1(\tau))| | \mathcal{F}_t \right].$$

Thus we have

$$|U(t) - U^m(t)| \leq \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} E^* \left[e^{-\int_t^\tau r(s) ds} |\psi(S^m(\tau)) - \psi(S_1(\tau))| | \mathcal{F}_t \right]. \tag{3.8}$$

Since, the set

$$\left\{ E^* \left[e^{-\int_t^\tau r(s) ds} |\psi(S^m(\tau)) - \psi(S_1(\tau))| | \mathcal{F}_t \right] : \tau \in \mathcal{T}_{t,T} \right\}$$

is directed (the maximum of two elements of the set is also an element of the set (see Dellacherie and Meyer, 1980, Chapters V–VIII, Appendix I-22, p. 431)), we have

$$\begin{aligned} E^* [|U(t) - U^m(t)|] \\ \leq \sup_{\tau \in \mathcal{T}_{t,T}} E^* \left[e^{-\int_t^\tau r(s) ds} |\psi(S^m(\tau)) - \psi(S_1(\tau))| \right] \\ \leq E^* \left[\sup_{0 \leq u \leq T} |\psi(S^m(u)) - \psi(S_1(u))| \right]. \end{aligned} \quad (3.9)$$

In view of the Proposition 3.2 and because of the uniform continuity of ψ , $\sup_{0 \leq u \leq T} |\psi(S^m(u)) - \psi(S_1(u))|$ converges to 0 in probability; furthermore, this sequence is uniformly integrable with respect to m (it can easily be seen using (1.7), (3.4) and the fact that $|\psi(x)| \leq Ax + B$ for every $x \in \mathbb{R}^+$).

Thus $\lim_{m \rightarrow +\infty} E^* [|U(t) - U^m(t)|] = 0$. \square

Remark 3.1. Theorem 3.1 implies that $U^m(0)$ converges to $U(0)$.

Remark 3.2. Suppose that ψ is a Lipschitz function (in the most common cases this is true), with Lipschitz constant L . If

$$C_m = \max \left\{ \sup_{s \in [0, T]} |\sigma_1(s) - \sigma^m(s)|; \sup_{s \in [0, T]} |f_1(s) - f^m(s)| \right\}$$

using (3.9) and (3.7), it is obvious that

$$E^* [|U(t) - U^m(t)|] \leq O(C_m).$$

If the coefficients are also Lipschitz functions and if Δ_m is the width of the partition, then

$$E^* [|U(t) - U^m(t)|] \leq O(\Delta_m).$$

Remark 3.3. It can easily be seen that the convergence is in $L^p(\mathbb{P}^*)$ with $p \in [1, +\infty)$. It could also be proved that we actually have the same convergence as that in Proposition 3.2, i.e. the paths of $U^m(t)$ converge to the paths of $U(t)$ uniformly in probability.

3.2. Discrete case

Let us consider the discrete time process $S^m(t_j)$ with $t_j \in \mathcal{P}^m$. This process is obtained by (2.1) with $N = 2^m$, $\sigma_j = \sigma(t_{j-1})$, $\phi_j = \phi(t_{j-1})$.

Let us consider the Snell envelope of $R(t_j)\psi(S^m(t_j))$:

$$\tilde{U}_j^m = \text{ess sup}_{\tau^m \in \mathcal{T}_{t_j, T}^m} E^* [R(\tau^m)\psi(S^m(\tau^m)) | \mathcal{F}_j],$$

where $\mathcal{T}_{t_j, T}^m$ is the set of stopping times with respect to the filtration $(\mathcal{F}_j)_{j=0, 1, \dots, 2^m}$ with $\mathcal{F}_j = \mathcal{F}_{t_j}$ and with values in $\{t_j, t_{j+1}, \dots, t_{2^m}\}$.

We set $U_D^m(t_j) = \tilde{U}_j^m[R(t_j)]^{-1}$ and consider right-continuous versions of $U(t)$ and $U^m(t)$.

Theorem 3.3. *Let $t \in \bigcup_m \mathcal{P}^m$. Then we have*

$$\lim_{m \rightarrow +\infty} E^* [|U_D^m(t) - U(t)|] = 0$$

uniformly with respect to $t \in \bigcup_m \mathcal{P}^m$.

Proof. For every $\tau \in \mathcal{T}_{0,T}$, we consider the discretization $d_m(\tau)(\omega) = \sum_{j=1}^{2^m} t_j I_{(t_{j-1}, t_j]}(\tau(\omega))$. It is easy to prove that $d_m(\tau) \in \mathcal{T}_{0,T}^m$.

Furthermore, if $\tau^m \in \mathcal{T}_{0,T}^m$, then $\tau^m \in \mathcal{T}_{0,T}$ and $d_m(\tau^m) = \tau^m$.

Thus

$$d_m(\mathcal{T}_{0,T}) = \mathcal{T}_{0,T}^m. \quad (3.10)$$

$S^m(d_m(\tau))$ converges to $S_1(\tau)$ in probability uniformly with respect to $\tau \in \mathcal{T}_{0,T}$. In fact

$$|S^m(d_m(\tau)) - S_1(\tau)| \leq |S_1(\tau) - S_1(d_m(\tau))| + \sup_{0 \leq s \leq T} |S^m(s) - S_1(s)|$$

and it can easily be shown that

$$|S_1(\tau) - S_1(d_m(\tau))| = \left| \int_{\tau}^{d_m(\tau)} S_1(s^-) \{ (b_1(s) - \theta(s)\sigma_1(s)) ds + \sigma_1(s) dW^*(s) + \phi_1(s) dN(s) \} \right| \quad (3.11)$$

converges to zero in $L^2(\mathbb{P}^*)$ uniformly with respect to $\tau \in \mathcal{T}_{0,T}$ (because of the fact that $|d_m(\tau) - \tau| \leq \Delta_m$ and the fact that $\sup_{0 \leq t \leq T} |S_1(t)| \in L^2(\mathbb{P}^*)$, see (1.7)). So we obtain, by Proposition 3.2, that $S^m(d_m(\tau))$ converges to $S_1(\tau)$ in probability uniformly with respect to $\tau \in \mathcal{T}_{0,T}$.

For the uniform continuity of ψ we have that

$$e^{-\int_{\tau}^{d_m(\tau)} r(s) ds} \psi(S^m(d_m(\tau)))$$

converges to

$$e^{-\int_{\tau}^{\tau} r(s) ds} \psi(S_1(\tau))$$

in probability uniformly with respect to $\tau \in \mathcal{T}_{0,T}$. Furthermore, the sequence

$$|e^{-\int_{\tau}^{d_m(\tau)} r(s) ds} \psi(S^m(d_m(\tau))) - e^{-\int_{\tau}^{\tau} r(s) ds} \psi(S_1(\tau))|$$

is uniformly integrable with respect to m and $\tau \in \mathcal{T}_{0,T}$ (recall (1.7), (3.4) and the fact that $|\psi(x)| \leq Ax + B$).

Thus

$$\lim_{m \rightarrow +\infty} E^* \left[\left| e^{-\int_t^{d_m(\tau)} r(s) ds} \psi(S^m(d_m(\tau))) - e^{-\int_t^\tau r(s) ds} \psi(S_1(\tau)) \right| \right] = 0 \quad (3.12)$$

uniformly with respect to $\tau \in \mathcal{T}_{0,T}$.

Considering (3.10), we obtain, in the same way as (3.8)

$$\begin{aligned} |U(t) - U_D^m(t)| &= \left| \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} E^* \left[e^{-\int_t^{d_m(\tau)} r(s) ds} |\psi(S^m(d_m(\tau)))| \mathcal{F}_t \right] \right. \\ &\quad \left. - \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} E^* \left[e^{-\int_t^\tau r(s) ds} |\psi(S_1(\tau))| \mathcal{F}_t \right] \right| \\ &\leq \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} E^* \left[\left| e^{-\int_t^{d_m(\tau)} r(s) ds} \psi(S^m(d_m(\tau))) - e^{-\int_t^\tau r(s) ds} \psi(S_1(\tau)) \right| \mathcal{F}_t \right]. \end{aligned}$$

Thus, because the set

$$\left\{ E^* \left[\left| e^{-\int_t^{d_m(\tau)} r(s) ds} \psi(S^m(d_m(\tau))) - e^{-\int_t^\tau r(s) ds} \psi(S_1(\tau)) \right| \mathcal{F}_t \right] : \tau \in \mathcal{T}_{t,T} \right\}$$

is directed, we have

$$\begin{aligned} E^* [|U(t) - U_D^m(t)|] \\ \leq \sup_{\tau \in \mathcal{T}_{t,T}} E^* \left[\left| e^{-\int_t^{d_m(\tau)} r(s) ds} \psi(S^m(d_m(\tau))) - e^{-\int_t^\tau r(s) ds} \psi(S_1(\tau)) \right| \right]. \end{aligned} \quad (3.13)$$

Thus, using (3.12), we conclude the proof. \square

If $t \notin \bigcup_m \mathcal{P}^m$, we can consider the sequence $\{t^m\}_m$, where $t^m \in \mathcal{P}^m$ and $t^m = \sum_{j=1}^{2^m} t_j I_{(t_{j-1}, t_j]}(t)$. Obviously t^m converges to t from the right. Since we consider a right-continuous version of $U(t)$, $\lim_{m \rightarrow +\infty} U(t^m) = U(t)$ a.s.. Since $|U(t^m) - U(t)|$ is uniformly integrable we have

$$\lim_{m \rightarrow +\infty} E^* [|U(t^m) - U(t)|] = 0,$$

i.e. there exists an $\bar{m} \in \mathbb{N}$ such that, for $m > \bar{m}$

$$E^* [|U(t^m) - U(t)|] < \frac{\varepsilon}{2}. \quad (3.14)$$

Furthermore, from Theorem 3.3, there exists an $\hat{m} > \bar{m}$ such that for every $m > \hat{m}$

$$E^* [|U(t^m) - U_D^m(t^m)|] < \frac{\varepsilon}{2}. \quad (3.15)$$

By (3.15) and (3.14) we have that, for every $m > \hat{m}$,

$$E^* [|U(t) - U_D^m(t^m)|] < \varepsilon. \quad (3.16)$$

Then the American option price may be approximated in $L^1(\mathbb{P}^*)$ with the value of a random variable that can be computed exactly (see Section 2). In particular, $U_D^m(0)$ converges to the American option price at time $t = 0$.

Remark 3.4. Suppose that ψ is a Lipschitz function with Lipschitz constant L . If $t \in \bigcup_m \mathcal{P}^m$, using (3.11), (3.7) and observing that $E^* [|S_1(\tau) - S_1(d_m(\tau))|] \leq C(\Delta_m)^{1/2}$, it is easy to conclude that

$$E^* [|U_D^m(t) - U(t)|] \leq O(\max\{C_m; \Delta_m^{1/2}\}).$$

If $t \notin \bigcup_m \mathcal{P}^m$ and if we consider the sequence t_m defined in the proof of Theorem 3.2, it can be shown that the speed of convergence of $U(t_m)$ to $U(t)$ in L^1 is at most $O(\max\{C_m; \Delta_m^{1/2}\})$. Then proceeding as we have done at the end of Section 3, we can conclude that, for every $t \in [0, T]$,

$$E^* [|U(t) - U_D^m(t^m)|] \leq O(\max\{C_m; \Delta_m^{1/2}\}).$$

If the coefficients are also Lipschitz functions, the speed of convergence is at most $O(\Delta_m^{1/2})$.

4. Almost sure approximations of the prices

4.1. Continuous case

Due to the Markovian properties of the model, we have $U(t) = u^*(t, S_1(t))$, with

$$u^*(t, x) = \sup_{\tau \in \mathcal{T}_{t,T}} E^* \left[e^{-\int_t^\tau r(s) ds} \psi(S_1^{t,x}(\tau)) \right]$$

and $U^m(t) = u_m^*(t, S^m(t))$, with

$$u_m^*(t, x) = \sup_{\tau \in \mathcal{T}_{t,T}} E^* \left[e^{-\int_t^\tau r(s) ds} \psi(S_{t,x}^m(\tau)) \right],$$

where $S_1^{t,x}(s)$, $t \leq s \leq T$, is the solution of (1.2), such that $S_1^{t,x}(t) = x$ and $S_{t,x}^m(s)$ is the solution of (3.5) such that $S_{t,x}^m(t) = x$ (see Jaillet et al., 1990, p. 265 or Zhang, 1994, p. 21, for a detailed proof).

We have

Theorem 4.1. For every $t \in [0, T]$, $u_m^*(t, x)$ converges to $u^*(t, x)$.

Proof. It is obvious from Theorem 3.1 and Remark 3.1. \square

Thus $u_m^*(t, S_1(t))$ converges to $U(t)$ a.s..

4.2. Discrete case

We also have that $U_m^D(t_j) = u_m^D(t_j, S^m(t_j))$, with

$$u_m^D(t_j, x) = \sup_{\tau^m \in \mathcal{T}_{t_j, t}^m} E^* \left[e^{-\int_{t_j}^{\tau^m} r(s) ds} \psi(S_{t_j, x}^m(\tau^m)) \right],$$

where $S_{t_j, x}^m(t_i)$, $t_i \geq t_j$, is the solution of (2.1) with $N = 2^m$, $\sigma_j = \sigma_1(t_{j-1})$, $\phi_j = \phi_1(t_{j-1})$ and such that $S_{t_j, x}^m(t_j) = x$.

We have

$$u_m^D(t_j, x) = \exp \left\{ \int_0^{t_j} r(s) ds \right\} u(j, x),$$

where $u(j, x)$ is defined by (2.3); the following result is obvious from Theorem 3.3:

Theorem 4.2. *Let $t \in \bigcup_m \mathcal{P}^m$. Then $u_m^D(t, x)$ converges to $u^*(t, x)$.*

This implies that $u_m^D(t, S_1(t))$ converges to $U(t)$ a.s..

If $t \notin \bigcup_m \mathcal{P}^m$, considering the sequence $\{t^m\}_m$ defined at the end of Section 3, we have that \hat{m} exists such that for every $m \geq \hat{m}$, $|u_m^D(t^m, S_1(t^m)) - U(t)| < \varepsilon$ a.s..

Remark 4.1. If we consider the convergence results studied in Theorems 4.1 and 4.2, we have, respectively, the same speed of convergence orders as the ones presented in Remarks 3.2 and 3.4.

5. The American put option

The put option is characterized by $\psi(x) = (K - x)^+$, where K is the exercise price.

Consider the case of a model with constant coefficients, i.e. suppose $\sigma_i(t) = \sigma_i$, $\phi_i(t) = \phi_i$, with $i = 1, 2$, and $r(t) = r$. Since we will work just with the first asset we put $\sigma_1(t) = \sigma$, $\phi_1(t) = \phi$ and $f_1(t) = f$. Thus $\sigma_m(t) = \sigma$, $\phi_m(t) = \phi$ and $u^*(t, x) = u_m^*(t, x)$.

We want to give an analytic formula for the computation of the American put prices. We will compute $u_m^D(t, x)$ for an approximation of $u^*(t, x)$.

Observe that for all x and $t \in \bigcup_m \mathcal{P}^m$,

$$0 \leq u^*(t, x) - u_m^D(t, x) \leq K(1 - e^{-r\Delta t m}),$$

where $r = \max\{r(s): 0 \leq s \leq T\}$.

This is true in view of Theorem 5 in (Carverhill and Webber, 1990, p. 89). In fact, if $t \in \bigcup_m \mathcal{P}^m$, the proof of this theorem presented in Carverhill and Webber (1990), may be adapted to our model.

Thus the approximation is very interesting because, if the annual interest rate is 10% (so that $r = \ln(11/10)$), the approximation for a nine months option with three quarterly exercise opportunities has error at most 2.5% of the exercise price.

The approach is analogous to that presented in Carverhill and Webber (1990), Geske (1977) and Geske (1979a, b).

In order to help the presentation we assume that the present time t is 0. We refer to Section 2. We put $q = 2^m$, the cardinality of \mathcal{P}^m . If $x_m(k)$ is the critical price at time $t_k \in \mathcal{P}^m$ for the discretized case, we have, being $t_q = T$,

$$u_m^D(T, x) = (K - x)^+ \quad \text{and} \quad x_m(q) = K.$$

If we consider $t_{q-1} \in \mathcal{P}^m$ we have

$$u_m^D(t_{q-1}, x) = (K - x)^+ I_{\{x \leq x_m(q-1)\}} + \hat{P}u_m^D(T, x) I_{\{x > x_m(q-1)\}},$$

where $\hat{P}u_m^D(T, x)$ is the price of an European put option starting at time t_{q-1} , with exercise time T . Thus adapting the computations presented in Theorem 2.1 of Mercurio and Runggaldier (1993) to the put case (see also Section 2), we obtain

$$\hat{P}u_m^D(T, x) = Kw_q^2 - xw_q^1,$$

where, if $\Lambda_m = p\lambda\Delta_m$,

$$w_q^2 = e^{-r\Delta_m} \sum_{h \in \mathbb{N}} e^{-\Lambda_m} \frac{\Lambda_m^h}{h!} N_1(d_{q-1,q}^2(h), 1)$$

and

$$w_q^1 = e^{-\phi\Delta_m} \sum_{h \in \mathbb{N}} e^{-\Lambda_m} \frac{\Lambda_m^h}{h!} e^{fh} N_1(d_{q-1,q}^1(h), 1),$$

with

$$d_{q-1,q}^1(h) = - \frac{\ln(x/x_m(q)) + (r + \frac{1}{2}\sigma^2)\Delta_m - \phi\Lambda_m + fh}{\sigma\Lambda_m^{1/2}}$$

and

$$d_{q-1,q}^2(h) = d_{q-1,q}^1(h) + \sigma\Delta_m^{1/2}.$$

The critical price $x_m(q-1)$, is obtained solving the equation $\hat{P}u_m^D(T, x) = (K - x)$. Since, if $t_k \in \mathcal{P}^m$,

$$u_m^D(t_k, x) = (K - x)^+ I_{\{x \leq x_m(k)\}} + \hat{P}u_m^D(t_{k+1}, x) I_{\{x > x_m(k)\}},$$

then continuing inductively backwards, we obtain

$$\hat{P}u_m^D(t_{k+1}, x) = Kw_{k+1}^2 - xw_{k+1}^1,$$

$$w_{k+1}^2 = \sum_{j=k+1}^q e^{-r(j-k)\Delta_m} \mathcal{P}_{j-k}(N_{j-k}(d_{k,j}^2, \mathbf{R}_{kj}))$$

and

$$w_{k+1}^1 = e^{-\phi(j-k)\Delta_m} \sum_{j=k+1}^q \mathcal{P}_{j-k}(e^{f(Q_{k+1} + \dots + Q_j)} N_{j-k}(d_{k,j}^1, \mathbf{R}_{kj})),$$

where $N_{j-k}(\cdot, R)$ is the multinormal distribution with dimension $j - k$ and covariance matrix R , and

$$\begin{aligned} d_{k,j}^1 &= (d_{k,k+1}, \dots, d_{k,j-1}, -d_{k,j}), \\ d_{k,j}^2 &= d_{k,j}^1 - \sqrt{\Delta_m} \sigma (1, \sqrt{2}, \dots, \sqrt{j-k-1}, -\sqrt{j-k}), \end{aligned}$$

where

$$d_{k,l} = \frac{\ln(x/x_m(l)) + (r + \frac{1}{2}\sigma^2)(l-k)\Delta_m - \phi\Delta_m(l-k) + f(Q_{k+1} + \dots + Q_l)}{\sigma\sqrt{\Delta_m(l-k)}}$$

and the covariance matrix R_{kj} is

$$R_{kj} = \begin{pmatrix} I & +r_{k+1,k+2} & \cdot & \cdot & \cdot & +r_{k+1,j-1} & -r_{k+1,j} \\ +r_{k+1,k+2} & 1 & & & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & \cdot \\ +r_{k+1,j-1} & & & & & \cdot & \cdot \\ -r_{k+1,j} & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix},$$

with $r_{p,q} = \left(\frac{p-k}{q-k}\right)^{1/2}$.

$\mathcal{P}_{j-k}(g(Q_{k+1}, \dots, Q_j))$ is the expected value of $g(Q_{k+1}, \dots, Q_j)$, where Q_r is an independent random variable with a Poisson distribution with parameter Δ_m . Thus

$$\begin{aligned} &\mathcal{P}_{j-k}(g(Q_{k+1}, \dots, Q_j)) \\ &= \sum_{q_{k+1}, \dots, q_j=0}^{+\infty} e^{-\Delta_m(j-k)} \frac{\Delta_m^{q_{k+1} + \dots + q_j}}{(q_{k+1})! \dots (q_j)!} g(q_{k+1}, \dots, q_j). \end{aligned}$$

Since this infinite sum may be truncated at a sufficiently large integer and since at each step k the critical price $x_m(k)$, is obtained solving the equation $\hat{P}u_m^D(t_{k+1}, x) = (K - x)$, the American put price is really computable.

For example, it is interesting to compare the Black–Scholes model with the model with jumps. In computing the relative prices, we have to observe that the presence of jumps modifies the observable volatility (see Zhang, 1994, Chapter 6) and that the change of measure from \mathbb{P} to \mathbb{P}^* implies the presence of the term $-p(t)\lambda(t)dt$ in (1.5).

Making computations varying the number of time steps, it can be observed that the choice of two time steps is good. If we take $r = 0.05$, $\sigma = 0.2$, $K = 45$, $x = 45$, $T = 3$, we obtain that the price in the Black–Scholes case is 3.5. Table 1 shows the American put price in some particular cases.

Table 1

	$\phi = 0.02$	$\phi = 0.04$	$\phi = -0.02$	$\phi = -0.04$
$\lambda = 1$	2.8	2.2	4.4	4.8
$\lambda = 0.5$	3.1	2.8	3.9	4.4

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